Threefold Way for Typical Entanglement (arXiv:2410.11309)

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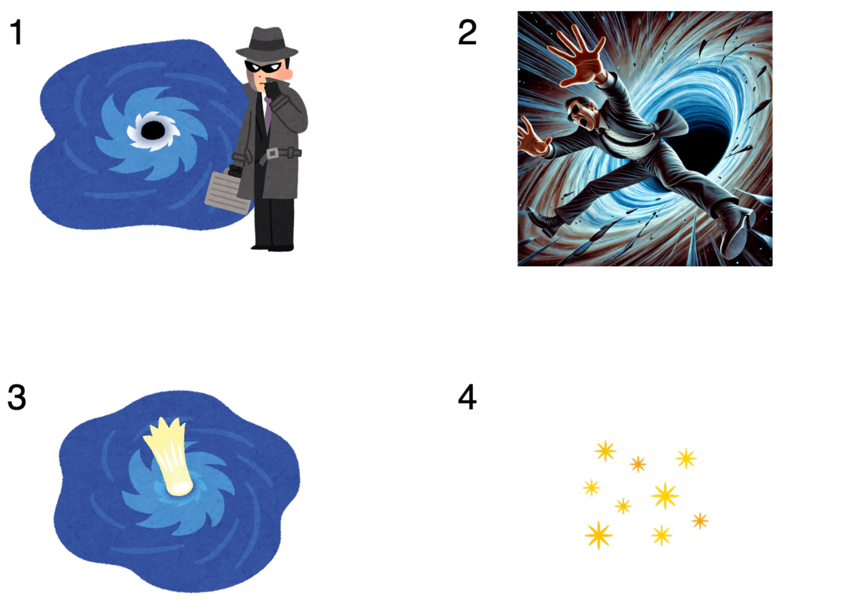
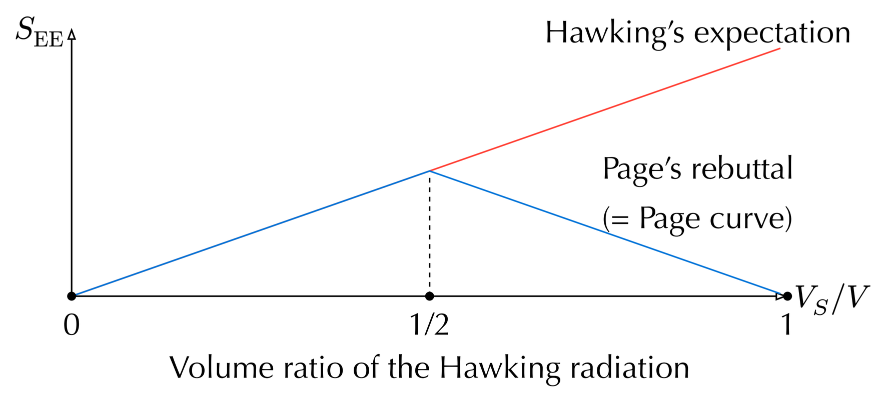
1. The Black Hole Information Paradox

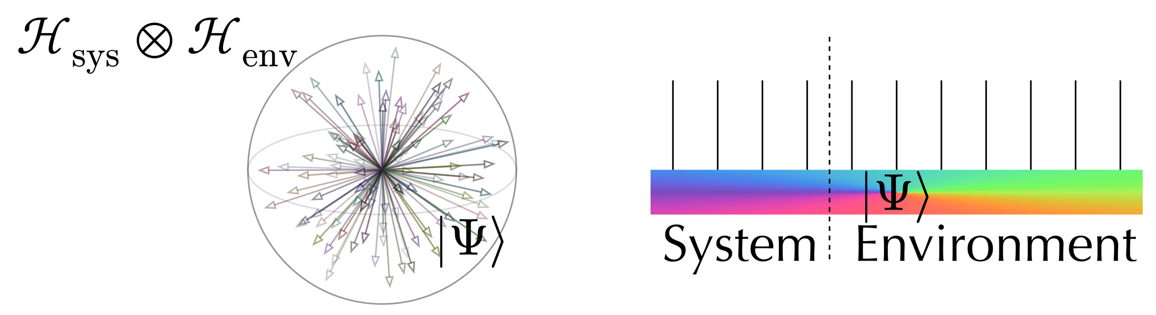
Thermalization is inconsistent (?) with unitary dynamics.





The entire dynamics should be **unitary** and **maximally chaotic**.
$\Rightarrow$ $|\Psi\rangle \sim U|0\rangle$, where U is a random unitary matrix.
This motivates the study of ***typical entanglement***.

2. Typical Entanglement



- **Typical** $\simeq$ Haar-random (uniformly random sampling on the entire Hilbert space)
- **Entanglement spectrum** = eigvals of the reduced density matrix
- Random matrix theory is useful to evaluate typical entanglement[1–4].

3. Laguerre Unitary Ensemble of Random Matrices

Preparing Haar-random state from the entire Hilbert space:

$$|\Psi\rangle \sim \text{Haar} \Leftrightarrow \rho_{\mathrm{sys}} = W\, W^{\dagger}.$$

where i.i.d. $W_{\mathrm{sys,env}} \sim \mathcal{CN}(0,1)$, $|\Psi\rangle = \sum_{\mathrm{sys,env}} W_{\mathrm{sys,env}} |\mathrm{sys,env}\rangle$ [5,6].

$\Rightarrow$ ρ_{sys} follows the **Laguerre unitary ensemble (LUE)** of RMT.

The joint probability p distribution of eigenvalues $\{\lambda_i\}$ of ρ_{sys} is ***Laguerre distribution***:

$$p(\lambda_1, \ldots, \lambda_m) \propto \prod_{i=1}^{m} \lambda_i^{\frac{\beta}{2}(n-m+1)-1} e^{-\frac{\beta}{2}\lambda_i} \prod_{1 \le i < j \le m} |\lambda_i - \lambda_j|^{\beta}$$

with $\beta = 2$ for LUE.

4. TRS and Threefold Way

What if we change complex random variables to real random variables?

$\Rightarrow$ $|\Psi\rangle$ and ρ_{sys} obtain **time reversal symmetry** (TRS) of $\mathcal{T} = K$:

$$|\Psi\rangle \to \mathcal{T}|\Psi\rangle = K|\Psi\rangle = |\Psi\rangle,$$

$$\rho_{\mathrm{sys}} \to \mathcal{T}\rho_{\mathrm{sys}}\mathcal{T}^{-1} = K\rho_{\mathrm{sys}}K = \overline{\rho_{\mathrm{sys}}} = \rho_{\mathrm{sys}}.$$

Note that there is the other & nonequivalent kind of TRS!

Time Reversal Symmetry
Integer spin: $\mathcal{T}_+^2 = +\mathbb{1}$. ex. $\mathcal{T}_+ = K$
<u>Half-integer spin</u>: $\mathcal{T}_-^2 = -\mathbb{1}$. ex. $\mathcal{T}_- = \sigma_y K$

Imposing TRS: $\mathcal{T}\rho_{\mathrm{sys}}\mathcal{T}^{-1} = \rho_{\mathrm{sys}}$ allows threefold way of Laguerre ensemble:

- $\beta = 1$ (Laguerre orthgonal ensemble)
- $\beta = 4$ (Laguerre symplectic ensemble).

5. Prohibition of $\mathcal{T}_-^2 = -\mathbb{1}$ TRS Eigenstate

Kramers' theorem: $\mathcal{T}_-^2 = -\mathbb{1}$ TRS cannot have the eigenstate $\mathcal{T}_-|\Psi\rangle = |\Psi\rangle$.

Proof: $\mathcal{T}$ is anti-unitary, thus $\langle \mathcal{T}\_a | \mathcal{T}\_b \rangle = \overline{\langle a|b\rangle}$.

$$\langle \psi | \mathcal{T}\_\psi \rangle = \overline{\langle \mathcal{T}\_\psi | \mathcal{T}_-^2 \psi \rangle} = \langle \mathcal{T}_-^2 \psi | \mathcal{T}\_\psi \rangle = -\langle \psi | \mathcal{T}\_\psi \rangle$$

This implies $\langle \psi | \mathcal{T}\_\psi \rangle = 0$, thus $|\psi\rangle$ is orthogonal to $\mathcal{T}|\psi\rangle$. □

6. Key Questions

Threefold way of Laguerre Ensemble[7]

	$\mathcal{T}_+^2 = +\mathbb{1}$ ($\mathcal{T}_+ \simeq K$) $\beta = 1$	No TRS $\beta = 2$	$\mathcal{T}_-^2 = -\mathbb{1}$ ($\mathcal{T}_- \simeq \sigma_y K$) $\beta = 4$
Laguerre	**orthogonal**, **LOE**	**unitary**, **LUE**	**symplectic** ensemble **LSE**
$\mathcal{T}|\Psi\rangle = |\Psi\rangle$:	real vector	complex vector	*unknown*

1. **Possible to construct the pure state that shows $\rho_{\mathrm{sys}} \sim$ LSE ?**
2. **Beyond threefold way if general symmetries?**

7. Key Idea: Symmetry Fractionalization

Fractionalization of TRS

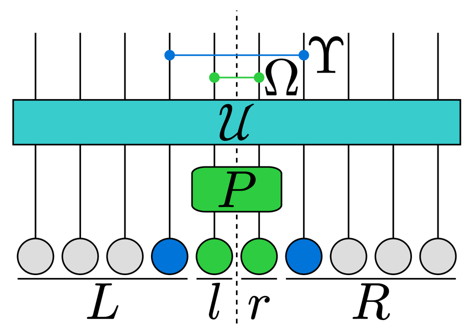
$$\mathcal{T}^2 = [\mathcal{T}_{\mathrm{sys}} \otimes \mathcal{T}_{\mathrm{env}}]^2 = +\mathbb{1}$$

$$\swarrow \quad \mathcal{T} = \Upsilon(\mathcal{T}_{\mathrm{sys}} \otimes \mathcal{T}_{\mathrm{env}})\Upsilon^{\dagger} \quad \searrow$$

$\mathcal{T}_{\mathrm{sys}}^2 = -\mathbb{1}$ $\qquad$ $\mathcal{T}_{\mathrm{env}}^2 = -\mathbb{1}$

- **We found $\Upsilon = \frac{1-i}{2}[\mathbb{1}_4 - i(\sigma_y \otimes \sigma_y)]$ and proved $\Upsilon|\Psi\rangle$ follows LSE.**
- Similarly, $\Omega = \sum_{g_l, g_r} \omega(g_r, g_r^{-1}g_l)|g_l, g_r\rangle\langle g_l, g_r|$ **fractionalizes general symmetries**: $\Omega[D(g) \otimes D(g)]\Omega^{\dagger} = \mathcal{D}(g) \otimes \overline{\mathcal{D}(g)}, \mathcal{D}(g)\mathcal{D}(g') = \omega(g, g')\mathcal{D}(gg')$.

8. General Setup



- $G = G_0$ or $G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$ (G_0: unitary, $\mathbb{Z}_2^{\mathcal{T}}$: anti-unitary).
- l, r: $|G_0|$-dimensional qudit (regular representation of G_0:$\langle g|g'\rangle = \delta_{g,g'}$).
- P: Projection of $|G_0|^2$-dimensional Hilbert space of $l \cup r$ onto G_0-symmetric $|G_0|$-dimensional basis $\forall g \in G_0$, $|\psi_g\rangle = \frac{1}{\sqrt{|G_0|}} \sum_{h \in G_0} |hg\rangle|h\rangle$.
- $\mathcal{U} \sim$ Haar measure on the projected $d_L d_R |G_0|$-dimensional space:

$$|\Psi\rangle = \sum_{L, g \in G_0, R} c_{L,g,R} |L\rangle|\psi_g\rangle|R\rangle = \frac{1}{\sqrt{|G_0|}} \sum_{L, g_l, g_r, R} c_{L, g_r^{-1}g_l, R} |L\rangle|g_l\rangle|g_r\rangle|R\rangle$$

- Ω fractionalizes G_0, Υ fractionalizes $\mathbb{Z}_2^{\mathcal{T}}$.

9. Results and Conclusions

Entanglement-spectrum statistics of…

$G = G_0$ is

$$\bigoplus_{\alpha} \left[\frac{\mathbb{1}_{d_\alpha}}{d_\alpha} \otimes \mathbf{LUE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha} \right].$$

$G = G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$ is

$$\left[\bigoplus_{\alpha : R_1} \frac{\mathbb{1}_{d_\alpha}}{d_\alpha} \otimes \mathbf{LOE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha} \right] \oplus \left[\bigoplus_{\alpha : R_0} \frac{\mathbb{1}_{2d_\alpha}}{d_\alpha} \otimes \mathbf{LUE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha} \right] \oplus \left[\bigoplus_{\alpha : R_{-1}} \frac{\mathbb{1}_{d_\alpha}}{d_\alpha} \otimes \mathbf{LSE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha} \right].$$

Until our work

- **The setup which follows LSE have been elusive.**

What this work reveraled are:

- **The LSE setup can be constructed by fractionalizing TRS** of the LOE setup.
- **Extended the setup to general symmetries.**
- Entanglement-spectrum statistics is **direct sum of the threefold way.**

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