Threefold Way for Typical Entanglement

Haruki Yagi[1], Ken Mochizuki[1,2], Zongping Gong[1]

1 Department of Applied Physics, Graduate School of Engineering, The University of Tokyo
2 Nonequilibrium Quantum Statistical Mechanics RIKEN Hakubi Research Team,RIKEN Cluster for Pioneering Research

arXiv:2410.11309

Keywords

Physics: typical entanglement, symmetry fractionalization, SPT phases
Mathematics: random matrices, unitary-antiunitary representations of groups

Background

Quantum Entanglement

Entanglement ≃ cannot be decomposed into tensor products of the local basis

$$\text{2 qubit:} \quad \frac{|0_A 0_B\rangle + |1_A 1_B\rangle}{\sqrt{2}} \quad (\neq |\psi_A\rangle \otimes |\psi_B\rangle)$$

$$\text{3 qubit:} \quad \frac{|0_A 0_B 0_C\rangle + |1_A 1_B 1_C\rangle}{\sqrt{2}}, \ \frac{|1_A 0_B 0_C\rangle + |0_A 1_B 0_C\rangle + |0_A 0_B 1_C\rangle}{\sqrt{3}}$$

$$\vdots$$

Entanglement entropy is a way to evaluate the strength of bipartite entanglement.

Partitioning the entire system into system (**sys**) and environment (**env**), we have

$$\rho_{\text{sys}} = \text{Tr}_{\text{env}} |\Psi\rangle\langle\Psi|, \quad S_{\text{EE}} = -\,\text{Tr}_{\text{sys}} \left[\rho_{\text{sys}} \ln \rho_{\text{sys}}\right].$$

The **larger S_{EE}**, the **more information is lost** in the environment.

The Black Hole Information Paradox

1

2

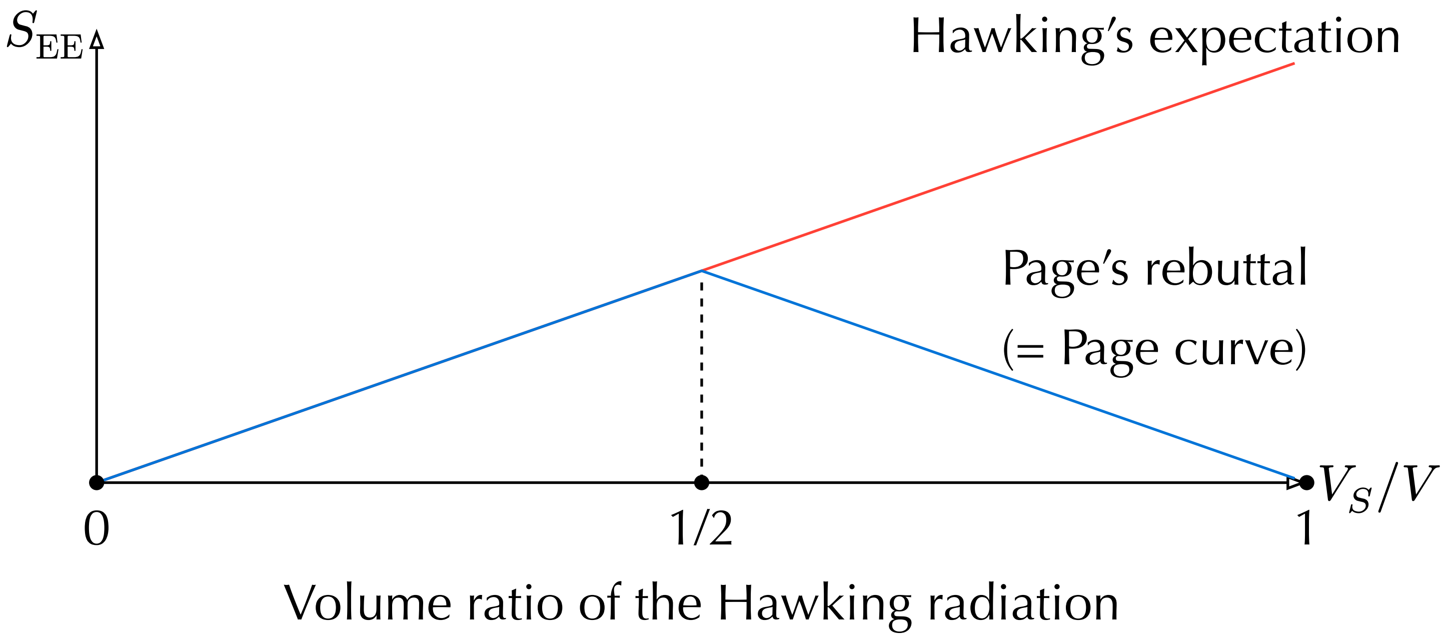
3

4

Page curve

Hawking: Information would be lost 😭 / Page: Information would be preserved 🧐

Page curve: **sys = the Hawking radiation, env = BH**

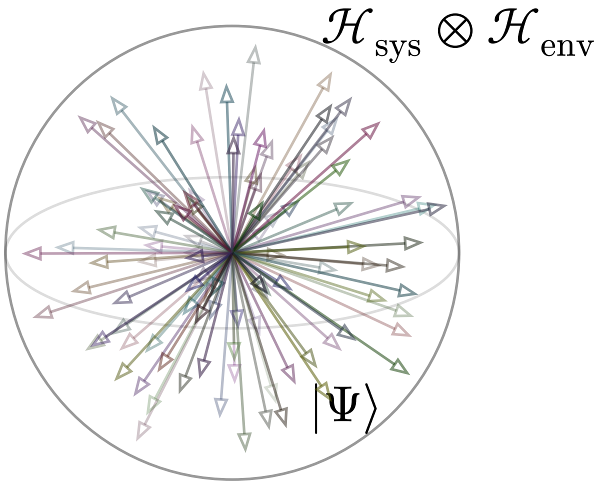


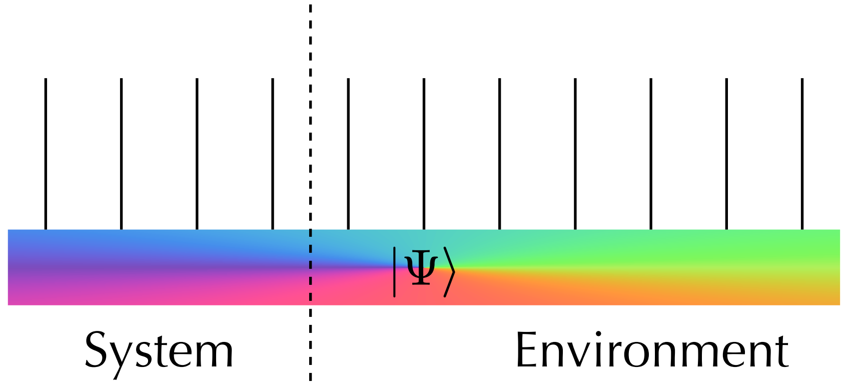
Page's consideration:

The entire dynamics should be **unitary** and **maximally chaotic**.

$\Rightarrow$ $|\Psi\rangle \sim U|0\rangle$, where U is a random unitary matrix.

Typical Entanglement





- **Typical** = **uniformly random** sampling on the entire Hilbert space $\sim$ Haar-random
- **Entanglement spectrum** = eigvals of the reduced density matrix $\rho_{\rm sys} = {\rm Tr}_{\rm env}|\psi\rangle\langle\psi|$
- **Typical entanglement** = entanglement of uniformly random-sampled state

The random matrix theory (RMT) is useful to evaluate typical entanglement. (Page, 1993; Sánchez-Ruiz, 1995; Sen, 1996).

Let's see how it works.

Laguerre Unitary Ensemble of Random Matrices

Preparing Haar-random pure state from the entire Hilbert space:

$$|\Psi\rangle = \sum_{\text{sys,env}} W_{\text{sys,env}} |\text{sys,env}\rangle, \quad \text{where} \quad \text{i.i.d. } W_{\text{sys,env}} \sim \mathcal{CN}(0,1).$$

$$|\Psi\rangle \sim \text{Haar} \Leftrightarrow \rho_{\text{sys}} = W \, W^\dagger \; .$$

(Zyczkowski and Sommers, 2001; Nechita, 2007)

$\Rightarrow \rho_{\text{sys}}$ follows the ***Laguerre unitary ensemble (LUE)*** of RMT.

The joint probability p distribution of eigenvalues $\{\lambda_i\}$ of ρ_{sys} is

$$p(\lambda_1, ..., \lambda_m) \propto \prod_{i=1}^{m} \lambda_i^{n-m} e^{-\lambda_i} \prod_{1 \le i < j \le m} |\lambda_i - \lambda_j|^2.$$

Time Reversal Symmetry and Threefold Way of Laguerre Ensemble

$$|\Psi\rangle = \sum_{\text{sys,env}} W_{\text{sys,env}} |\text{sys,env}\rangle, \quad \text{where} \quad \text{i.i.d. } W_{\text{sys,env}} \sim \mathcal{CN}(0,1)$$

What if we change *complex* random variables to *real* random variables?

$|\Psi\rangle$ and ρ_{sys} obtain **Time Reversal Symmetry** (TRS) of $\mathcal{T} = K$:

$$|\Psi\rangle \to \mathcal{T}|\Psi\rangle = K|\Psi\rangle = |\Psi\rangle,$$
$$\rho_{\text{sys}} \to \mathcal{T}\rho_{\text{sys}}\mathcal{T}^{-1} = K\rho_{\text{sys}}K = \overline{\rho_{\text{sys}}} = \rho_{\text{sys}}.$$

Note that there is **the other & nonequivalent kind of TRS!**

Time Reversal Symmetry

Integer spin: $\mathcal{T}_+^2 = +\mathbb{1}$. ex. $\mathcal{T}_+ = K$

Half-integer spin: $\mathcal{T}_-^2 = -\mathbb{1}$. ex. $\mathcal{T}_- = \sigma_y K$

Let's look at the eigenvalue statistics of the density matrix with TRS.

Time Reversal Symmetry and Threefold Way of Laguerre Ensemble

$$p(\lambda_1, ..., \lambda_m) \propto \prod_{i=1}^{m} \lambda_i^{\frac{\beta}{2}(n-m+1)-1} e^{-\frac{\beta}{2}\lambda_i} \prod_{1\le i<j\le m} |\lambda_i - \lambda_j|^\beta$$

No TRS: $\beta = 2.$

Imposing TRS: $\mathcal{T}\rho_{\text{sys}}\mathcal{T}^{-1} = \rho_{\text{sys}}$ allows $\beta = 1$ and/or $\beta = 4.$

Threefold way of Laguerre ensemble

	$\mathcal{T}_+^2 = +\mathbb{1}$ $(\mathcal{T}_+ \simeq K)$	No TRS	$\mathcal{T}_-^2 = -\mathbb{1}$ $(\mathcal{T}_- \simeq \sigma_y K)$	
	$\beta = 1$	$\beta = 2$	$\beta = 4$	
Laguerre	**orthogonal**,	**unitary**,	**symplectic**	ensemble
	LOE	**LUE**	**LSE**	

Kramers' theorem and prohibition of $\mathcal{T}_-^2 = -\mathbb{1}$ TRS eigenstate

TRS for ρ_{sys}: $\mathcal{T}\rho_{\text{sys}}\mathcal{T}^{-1} = \rho_{\text{sys}}$ always has solutions. However, TRS for $|\Psi\rangle$ is ill-defined:

Theorem: $\mathcal{T}_-^2 = -\mathbb{1}$ **TRS cannot have the eigenstate** $\mathcal{T}_-|\Psi\rangle = |\Psi\rangle$.

Proof: Time reversal operator $\mathcal{T}$ is anti-unitary, thus $\langle\mathcal{T}_-a|\mathcal{T}_-b\rangle = \overline{\langle a|b\rangle}$.

$$\langle\psi|\mathcal{T}_-\psi\rangle = \overline{\langle\mathcal{T}_-\psi|\mathcal{T}_-^2\psi\rangle} = \langle\mathcal{T}_-^2\psi|\mathcal{T}_-\psi\rangle = -\langle\psi|\mathcal{T}_-\psi\rangle$$

This implies $\langle\psi|\mathcal{T}_-\psi\rangle = 0$ for an arbitary state, thus $|\psi\rangle$ is orthogonal to $\mathcal{T}|\psi\rangle$. ■

	$\mathcal{T}_+^2 = +\mathbb{1}$ $(\mathcal{T}_+ \simeq K)$	No TRS	$\mathcal{T}_-^2 = -\mathbb{1}$ $(\mathcal{T}_- \simeq \sigma_y K)$	
	$\beta = 1$	$\beta = 2$	$\beta = 4$	
Laguerre	**orthogonal**, **LOE**	**unitary**, **LUE**	**symplectic** **LSE**	ensemble
$\mathcal{T}|\Psi\rangle = |\Psi\rangle$:	real vector	complex vector	*unknown!*	

Key Questions and Solutions

Q1. Possible to construct $\rho_{\mathrm{sys}} \sim$ LSE ?

Q2. Beyond threefold way if general symmetries?

Q1. Possible to construct $|\Psi\rangle$ whose $\rho_{\rm sys} \sim$ LSE ?

A1. Yes, by *fractionalization* of TRS.

Q2. Beyond threefold way if general symmetries?

A2. Never. Direct sum of threefold way.

1. Exploring the LSE-Realizing System

Requirement	$\mathcal{T}_+\rho_{\mathrm{sys}}\mathcal{T}_+^{-1} = \rho_{\mathrm{sys}}$	$\mathcal{T}_-\rho_{\mathrm{sys}}\mathcal{T}_-^{-1} = \rho_{\mathrm{sys}}$
Existence of pure state	Yes, $\mathcal{T}_+\lvert\Psi\rangle = \lvert\Psi\rangle$	**No**, $\mathcal{T}_-\lvert\Psi\rangle \neq \lvert\Psi\rangle$
Pure state	real vector	**???**

$$\Downarrow$$

Fractionalization of TRS

$$\mathcal{T}^2 = \left[\mathcal{T}_{\mathrm{sys}} \otimes \mathcal{T}_{\mathrm{env}}\right]^2 = +\mathbb{1}$$

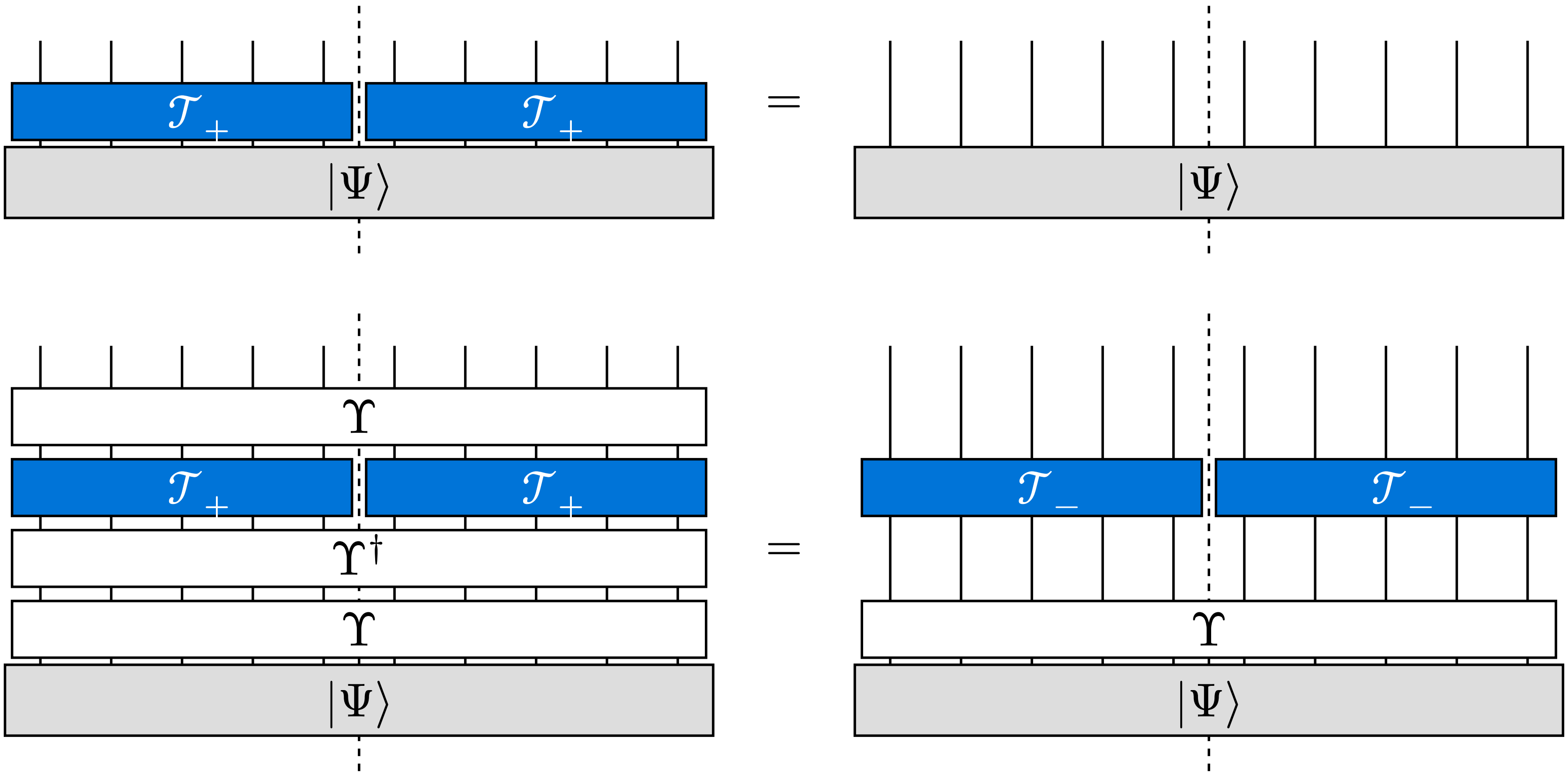
$$\swarrow \quad \mathcal{T} = \Upsilon\left(\mathcal{T}_{\mathrm{sys}} \otimes \mathcal{T}_{\mathrm{env}}\right)\Upsilon^\dagger \quad \searrow$$

$$\mathcal{T}_{\mathrm{sys}}^2 = -\mathbb{1} \qquad\qquad \mathcal{T}_{\mathrm{env}}^2 = -\mathbb{1}$$

- Υ fractionalizes TRS pure state of LOE $\lvert\Psi\rangle$.
- **We found $\Upsilon = \frac{1-\mathrm{i}}{2}\left[\mathbb{1}_4 - \mathrm{i}\sigma_y \otimes \sigma_y\right]$ and proved $\Upsilon\lvert\Psi\rangle$ follows LSE.**

1. Exploring the LSE-Realizing System

Fractionalization of TRS: $\Upsilon(\mathcal{T}_+ \otimes \mathcal{T}_+)\Upsilon^\dagger = \mathcal{T}_- \otimes \mathcal{T}_-$



2. Imposing General (Finite) Symmetries

- *Groups* (mathematics) describe symmetries.

Group G
Set with an assosiative multiplication. Identity and inverse exist for every elements.

  - Examples: $\mathbb{Z}_m,\ \mathbb{Z}_m \times \mathbb{Z}_n,\ C_{3v},\ Q_8,\ \mathbb{Z}_2^{\mathcal{T}}$ **(TRS)**, $\cdots$
- Constraints to the states / operators are given by *unitary-antiunitary representations*.

Unitary-antiunitary representation D of G
$D(a)D(b) = D(ab)$
for $\forall a, b \in G$.

- *Projective representations* describe *anomalous symmetries* and (1+1)D SPT phases.

Projective representation $\mathcal{D}$ of G
$\mathcal{D}(a)\mathcal{D}(b) = \omega(a,b)\mathcal{D}(ab)$
for $\forall a, b \in G,\ \omega : G \times G \to U(1)$.

2. Imposing General (Finite) Symmetries

Symmetry Fractionalization

$$\forall g, g' \in G,\ D(g)D(g') = D(gg') \ \text{(regular rep)}$$

$$\swarrow \quad \Omega(D \otimes D)\Omega^\dagger = \mathcal{D} \otimes \mathcal{D}' \quad \searrow$$

$\mathcal{D}(g)\mathcal{D}(g') = \omega(g, g')\mathcal{D}(gg')$	$\mathcal{D}'(g)\mathcal{D}'(g') = \overline{\omega(g, g')}\mathcal{D}'(gg')$

The eqiuvalence relation between reps is defined by phase modulation;

$$\mathcal{D}_{\text{new}}(g) = e^{\mathrm{i}\phi(g)}\mathcal{D}(g)$$

Then, reps are classified by **2nd order group cohomology $H^2(G, U(1))$.**

- Regular reps of different classes lead to different irreducible decompositions.

2. Imposing General (Finite) Symmetries

- Groups can be extended by *(semi)direct product*.

Direct product and semidirect product of groups

Direct: $G = G_1 \times G_2$, $(a_1, a_2)(b_1, b_2) = (a_1 b_1, a_2 b_2)$

Semidirect: $G = G_1 \rtimes G_2$, $(a_1, a_2)(b_1, b_2) = \big(a_1 f_{a_2}(b_1), a_2 b_2\big)$

- Any unitary-antiunitary reps can be decomposed into a set of *irreducible reps*.

Irreducible representations

Reps for which all the elements in a unitary transformation cannot be further decomposed into direct sums at the same time are called an irreducible reps.

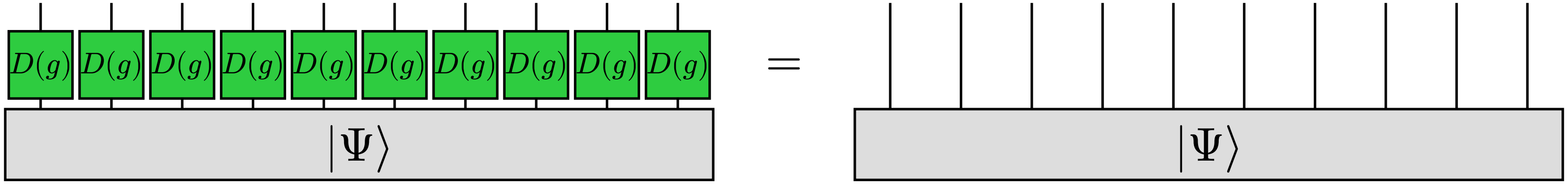
- *The Regular representation* can be chosen for the most natural rep.

Regular representation

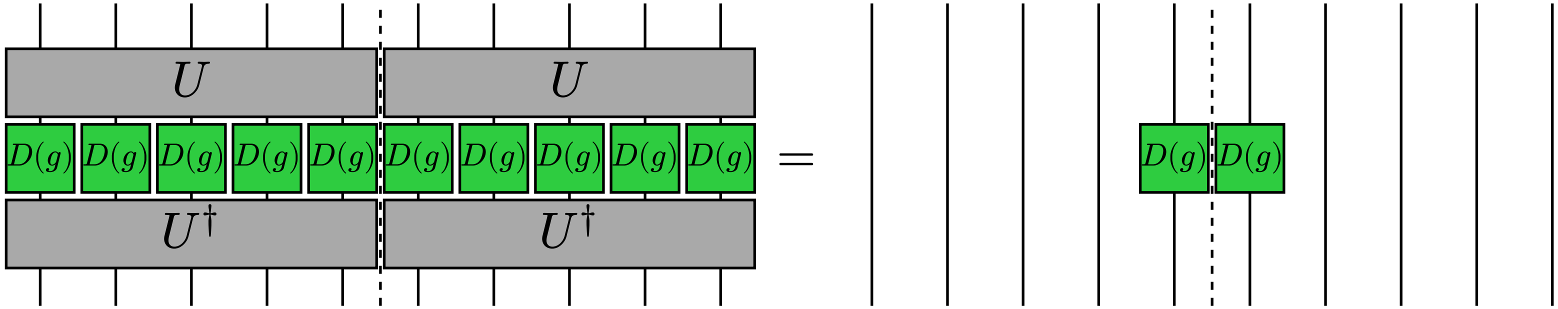
$D(g) = \big[\delta\big(g_i \; g \; g_j^{-1}\big)\big]$ where $\delta(e) = 1$ and otherwise $\delta(g) = 0$

2. Imposing General (Finite) Symmetries

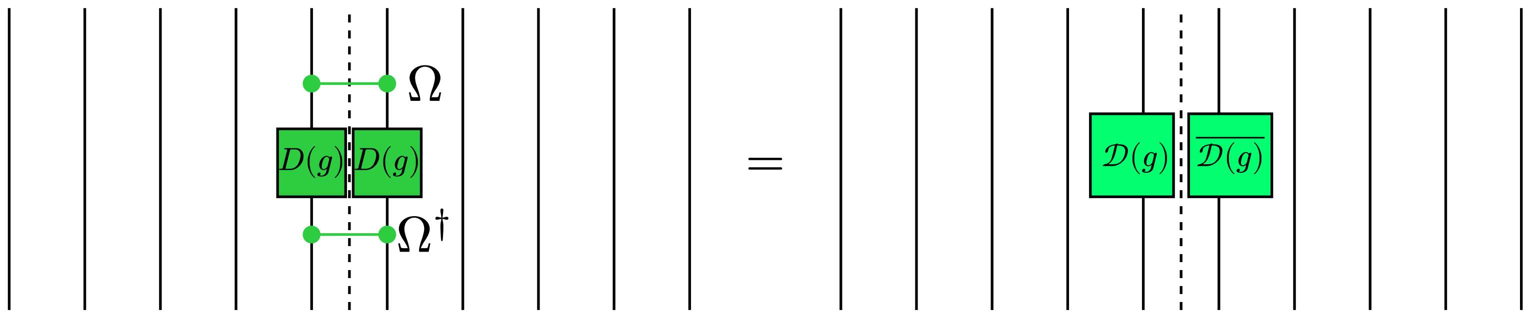
- We consider **random G-symmetric states $|\Psi\rangle$**, where $\boldsymbol{G = G_0}$ **or** $\boldsymbol{G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}}$.



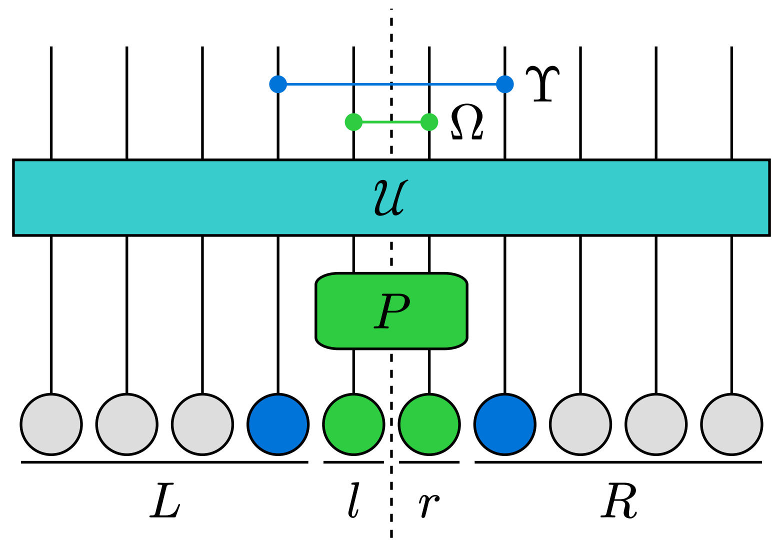
- We concentrate onsite symmetries s.t. the entanglement spectrum is not changed.



- Fractionalization of G_0 by Ω can be done independently of $\mathbb{Z}_2^{\mathcal{T}}$.



General Setup



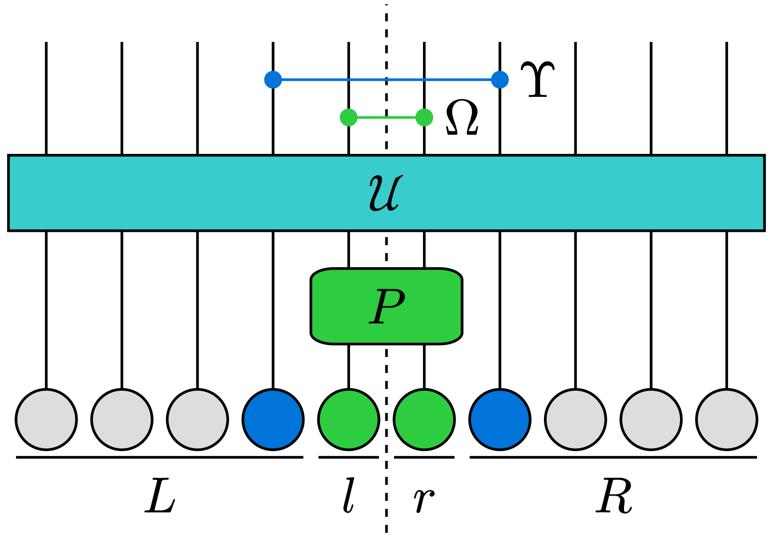
The considered symmetries are $G = G_0$ or $G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$ (G_0: unitary, $\mathbb{Z}_2^{\mathcal{T}}$: anti-unitary).

- $\boldsymbol{G_0}$: Green circles.
  - l, r: $|G_0|$-dimensional qudit (Finite. The regular rep of G_0: $\langle g|g'\rangle = \delta_{g,g'}$)
  - P: Projection of $|G_0|^2$-dim space $l \cup r$ onto G_0-symmetric $|G_0|$-dim basis below:

$$\forall g \in G_0,\ \left|\psi_g\right\rangle = \frac{1}{\sqrt{|G_0|}} \sum_{h \in G_0} |hg\rangle|h\rangle. \quad \left(D(g) \otimes D(g)\left|\psi_g\right\rangle = \left|\psi_g\right\rangle\right).$$

- $\boldsymbol{\mathbb{Z}_2^{\mathcal{T}}}$: Blue circles = 2-dimensional cuts of each subsystem (only when Υ is necessary).

General Setup



- **Randomness**: $\mathcal{U} \sim$ Haar measure on the projected $d_L d_R |G_0|$-dimensional space:

$$|\Psi\rangle = \sum_{L,g\in G_0,R} c_{L,g,R}|L\rangle\big|\psi_g\big\rangle|R\rangle = \frac{1}{\sqrt{|G_0|}} \sum_{L,g_l,g_r,R} c_{L,g_r^{-1}g_l,R}|L\rangle|g_l\rangle|g_r\rangle|R\rangle$$

- **Fractionalization**: Ω fractionalizes G_0, Υ fractionalizes $\mathbb{Z}_2^{\mathcal{T}}$:

$$\Omega = \sum_{g_l,g_r} \omega(g_r, g_r^{-1}g_l)|g_l, g_r\rangle\langle g_l, g_r|, \ \ \Upsilon = \frac{1-\mathrm{i}}{2}\big(\mathbb{1}_4 - \mathrm{i}\sigma_y \otimes \sigma_y\big).$$

Results and Conclusion

Direct Sum into the Threefold Way

Entanglement-spectrum statistics of $G = G_0$ is

$$\bigoplus_{\alpha}\left[\frac{\mathbb{1}_{d_\alpha}}{d_\alpha}\otimes \mathbf{LUE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha}\right],$$

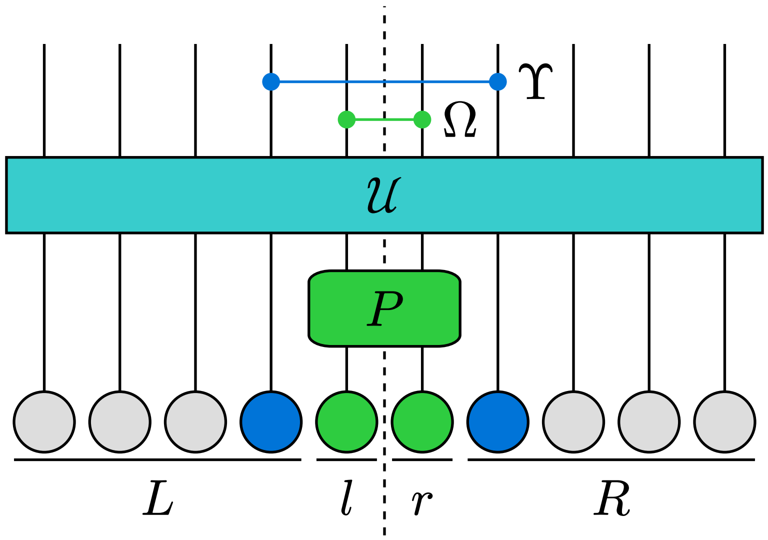
On the other hand, that of $G = G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$ is

$$\left[\bigoplus_{\alpha:R_+}\frac{\mathbb{1}_{d_\alpha}}{d_\alpha}\otimes \mathbf{LOE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha}\right]\oplus\left[\bigoplus_{\alpha:R_0}\frac{\mathbb{1}_{2d_\alpha}}{d_\alpha}\otimes \mathbf{LUE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha}\right]\oplus\left[\bigoplus_{\alpha:R_-}\frac{\mathbb{1}_{d_\alpha}}{d_\alpha}\otimes \mathbf{LSE}_{\alpha}^{d_L d_\alpha \times d_R d_\alpha}\right].$$

Entanglement-spectrum statistics of random symmetric states is
always able to be decomposed into the direct sum of the threefold way[1].

[1] This result is also the Laguerre version of Dyson's Gaussian threefold way (1962).

Conclusion



Until our work

- **The setup which follows LSE have been elusive.**

What this work reveraled are:

- **The LSE setup can be constructed by fractionalizing TRS** of the LOE setup.
- **Extended the setup to general symmetries.**
- Entanglement-spectrum statistics is **direct sum of the threefold way.**

References

Nechita, I. (2007) “Asymptotics of random density matrices,” in *Annales Henri Poincaré*, pp. 1521–1538

Page, D. N. (1993) “Average entropy of a subsystem,” *Phys. Rev. Lett.*, 71(9), pp. 1291–1294. Available at: https://doi.org/10.1103/PhysRevLett.71.1291

Sen, S. (1996) “Average Entropy of a Quantum Subsystem,” *Phys. Rev. Lett.*, 77(1), pp. 1–3. Available at: https://doi.org/10.1103/PhysRevLett.77.1

Sánchez-Ruiz, J. (1995) “Simple proof of Page's conjecture on the average entropy of a subsystem,” *Phys. Rev. E*, 52(5), pp. 5653–5655. Available at: https://doi.org/10.1103/PhysRevE.52.5653

Zyczkowski, K. and Sommers, H.-J. (2001) “Induced measures in the space of mixed quantum states,” *Journal of Physics A: Mathematical and General*, 34(35), p. 7111

Supplemental Materials

Some additional information on group reps

- An irrep is either one of *real, complex, or pseudoreal*.
- **We consider the cases** $\boldsymbol{G = G_0}$ **or** $\boldsymbol{G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}}$.
- For $G = G_0 \rtimes \mathbb{Z}_2^{\mathcal{T}}$, one can define *the indicator* to know irreps real, complex, or pseudoreal.

Indicator of irrep

$$\iota_\alpha = \frac{1}{|G_0|} \sum_{g \in G_0} \mathrm{Tr}[\mathcal{D}_\alpha(\tilde{g}) \mathcal{D}_\alpha(g)],$$

$$\iota = \begin{cases} 1 : \text{real} \\ 0 : \text{complex} \\ -1 : \text{pseudoreal} \end{cases}$$

The cocycle (=cohomology class) can be decoupled $\omega = \omega_{G_0} \omega_{\mathbb{Z}_2^{\mathcal{T}}}$.

- $R_\pm$ = set irreps that satisfies $\iota = \pm\omega(t,t)$. $\omega(t,t) = 1(-1)$ in the absent (present) of Υ.