Entanglement Spectrum Resolved by Loop Symmetries arxiv:2510.18350

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We present a rigorous and general analysis of quantum many-body entanglement possessing a $(d-2)$-form $\mathrm{Rep}(G)$ symmetry.

1. Context and Question

- (0-form) **Group** symmetries in quantum mechanics are either unitary or anti-unitary [Wigner 1931].
- Thus, operators are resolved into the direct sum of real, complex, or pseudoreal blocks: the **threefold way** [Dyson 1962].
- So are **reduced density matrices** [Yagi, Mochizuki and Gong 2024], but in a slightly different way…
- Topological wilson loops form $(d-2)$-form **non-invertible symmetries**. They can be obtained by **gauging** group symmetries, and higher categories are inevitable [Bhardwaj, Schafer-Nameki and Wu 2022].

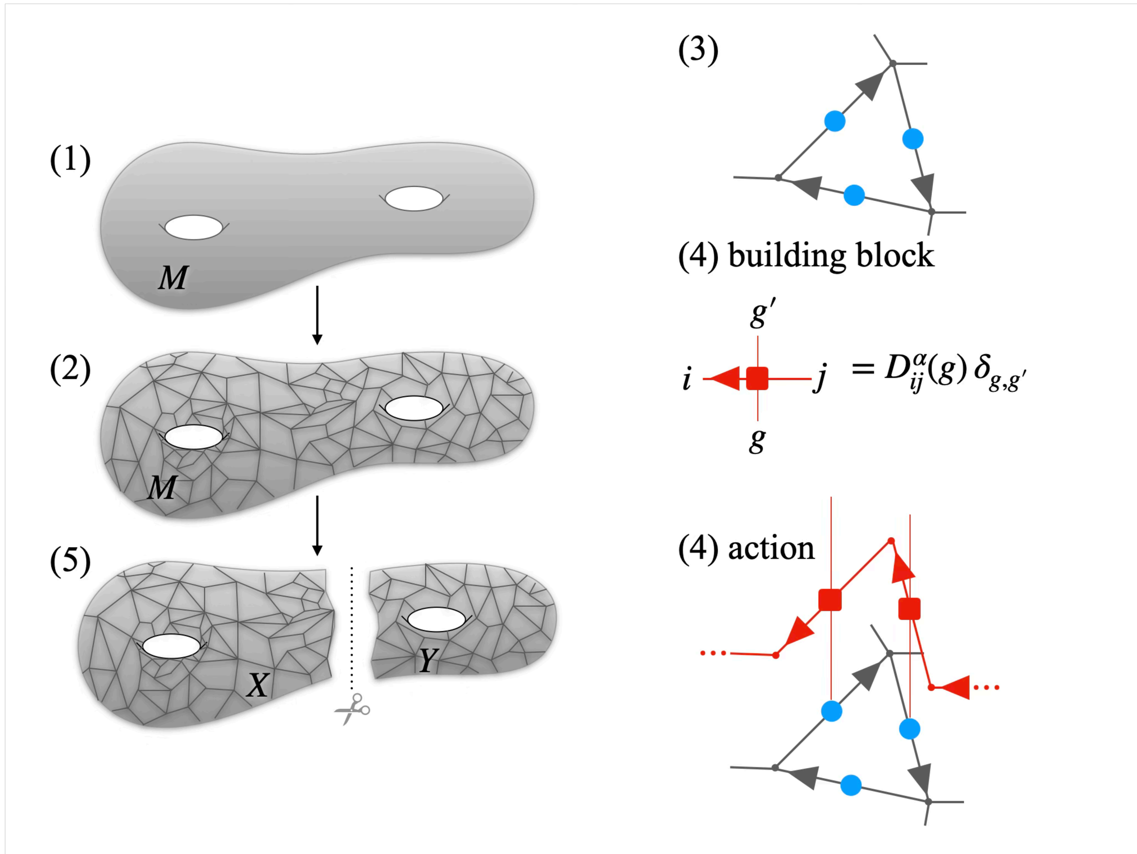
Question:

How do such non-invertible symmetries resolve the reduced density matrices?

2. Setup

Consider a general theory in which $(d-2)$-form $\mathrm{Rep}(G)$ symmetry can be defined:

1. Fix a space (not spacetime) manifold M and a finite group G.
2. Take a directed graph as a good discretization of M.
3. Place G-spins' Hilbert spaces on the edges.
4. Take the Hilbert subspace which respects $(d-2)$-form $\mathrm{Rep}(G)$ symmetry as the Wilson loop on M.
5. Split M and edges into X and Y and consider the entanglement between X and Y. Let $\partial := X \cap Y$.



The state can be expanded as $|\Psi\rangle = \sum_{\{g_X\},\{g_Y\}} \Psi_{\{g_X\},\{g_Y\}} |\{g_X\}\rangle|\{g_Y\}\rangle$.

We define a matrix $W := \sum_{\{g_X\},\{g_Y\}} \Psi_{\{g_X\},\{g_Y\}} |\{g_X\}\rangle\langle\{g_Y\}|$ which satisfies $\rho_X = WW^\dagger$, where ρ_X is the reduced density matrix on X. Hereafter, we focus on the structure of W.

3. Main Result 1: the Algorithm and the Block Structure

We use basic category theory to make the algorithm applicable to any manifold. **Many examples are shown in our paper.**

First, the split of a manifold M is represented as the pushout:

$$\begin{array}{ccc} M & \hookleftarrow & X \\ \uparrow & & \uparrow \\ Y & \hookrightarrow & \partial \end{array}$$

Put a base point on each connected component of ∂ for the next step. (We denote the set of base points as A.)

Next, applying the **Seifert-van Kampen theorem**, the commutative diagram of fundamental groupoids is again the pushout.

$$\begin{array}{ccc} \pi_1(M,A) & \xleftarrow{p_X} & \pi_1(X,A) \\ {\scriptstyle p_Y}\uparrow & & \uparrow{\scriptstyle i_X} \\ \pi_1(Y,A) & \xleftarrow{i_Y} & \pi_1(\partial,A) \end{array}$$

Lastly, since $\mathrm{Hom}(*,G)$ is contravariant, the commutative diagram becomes the pullback.

$$\begin{array}{ccc} \mathrm{Hom}(\pi_1(M,A),G) & \xrightarrow{\Pi_X} & \mathrm{Hom}(\pi_1(X,A),G) \\ {\scriptstyle \Pi_Y}\downarrow & & \downarrow{\scriptstyle r_X} \\ \mathrm{Hom}(\pi_1(Y,A),G) & \xrightarrow{r_Y} & \mathrm{Hom}(\pi_1(\partial,A),G) \end{array}$$

Theorem (Main result 1 of our work).

Define Im as the image in $\mathrm{Hom}(\pi_1(\partial,A),G)$ of the commutative diagram.

Then, the matrix W admits the following decomposition (omitting zeros).

$$W \in \left(\bigoplus_{\phi\in\mathrm{Im}} \mathbb{C}^{|r_X^{-1}(\phi)|\times|r_Y^{-1}(\phi)|} \right) \otimes \left(\bigoplus_{j=1}^{|G|^{|V_\partial|-|A|}} \mathbb{C}^{|G|^{|V_X|}\times|G|^{|V_Y|}} \right)$$

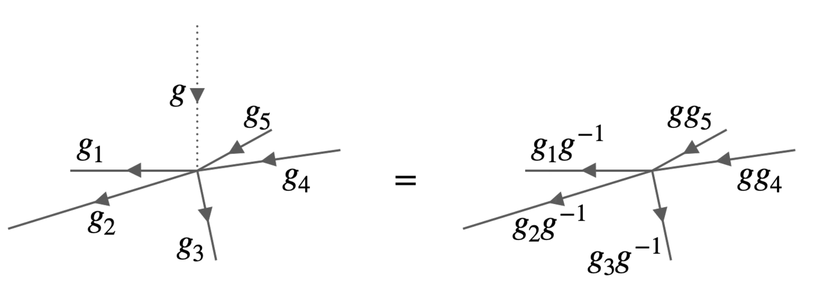
The former half reflects the twisted boundary condition and the latter half is geometric.

Discussion

- In contrast to the 0-form non-Abelian group symmetries, **degeneracies in the entanglement spectrum are not enforced** by $(d-2)$-form $\mathrm{Rep}(G)$ symmetries.
  - c.f. [Xu, Rakovszky, Knap, and Pollmann 2024]
- The symmetry enforces "**area-law entanglement spectrum fragmentation**". However, entanglement entropy can take **volume-law values**.
  - c.f. [Fukushima and Hamazaki 2023]

4. Main Result 2: Li-Haldane Correspondence for Kitaev Quantum Double Model

The Gauss law constraint can be incorporated as a further restriction. Our result can be refined by this constraint:



$$W \in \left(\bigoplus_{[\phi]\in\mathrm{Im}/G^{|A|}} \mathbb{I}_{|[\phi]|} \otimes \left(\bigoplus_{\alpha_{[\phi]}} \mathbb{C}^{x_{\alpha_{[\phi]}}\times y_{\alpha_{[\phi]}}} \otimes \mathbb{I}_{d_{\alpha_{[\phi]}}} \right) \right) \otimes \bigoplus_{j=1}^{|G|^{|V_\partial|-|A|}} \begin{pmatrix} 1 & \dots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \dots & 1 \end{pmatrix}.$$

Details (including the definitions of the symbols) can be found in our paper.

The key points are as follows:

- From this result, it is straightforward to show that this is an **area-law entangled** state.
- Entanglement spectrum degeneracy is restored by Gauss law, and it is $|[\phi]| \times d_{\alpha_{[\phi]}}$-fold degenerate for each sector.

Specifically for $\dim M = 2$, the model is known as the **Kitaev quantum double model**.

Our analysis yields the following (Main result 2 of our work, including reproduction of known results):

- **Entanglement blocks are labeled by the anyons** (of quantum double $D(G)$) transmitting through ∂.
  - This is consistent with the **Li-Haldane correspondence** (although the model is non-chiral).
- **Entanglement block size = Fusion multiplicities in X/Y**
- **Entanglement spectrum degeneracy = Product of quantum dimensions** $\times$ area-law fragments

5. Conclusion and Outlook

- A **rigorous, manifold-agnostic** analysis of quantum many-body entanglement possessing a $(d-2)$-form $\mathrm{Rep}(G)$ symmetry is carried out using basic category theory.
- We **clarified how non-invertible symmetries determine the structure of entanglement** through the presence or absence of entanglement spectrum degeneracy and sector decomposition.
- This framework is expected to be readily applicable to **multipartite entanglement**. It may also serve as a probe to extract **higher-order information** from entanglement in topological order.